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LexBFS-orderings and powers of chordal graphs[1]

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Abstract

For an undirected graph G the kth power G^k of G is the graph with the same vertex set as G where two vertices are adjacent iff their distance is at most k in G. In this paper we prove that any LexBFS-ordering of a chordal graph is a common perfect elimination ordering of all odd powers of this graph. Moreover, we characterize those chordal graphs by forbidden isometric subgraphs for which any LexBFS-ordering of the graph is a common perfect elimination ordering of all powers. For MCS-orderings of chordal graphs the situation is worse: even for trees MCS does not give a common perfect elimination ordering of powers.

1. Introduction

In the last years some papers investigating powers of chordal graphs were published. One of the first results in this field is due to Duchet [6]: If G^k is chordal then G^{k+2} is so. In particular, odd powers of chordal graphs are chordal, whereas even powers of chordal graphs are in general not chordal. Chordal graphs with chordal square were characterized by forbidden configurations in [10].

It is well-known that every chordal graph has a perfect elimination ordering. Thus each chordal power of an arbitrary graph has a perfect elimination ordering. A natural question is whether there is a common perfect elimination ordering of all (or some) chordal powers of a given graph. The first result in this direction using minimal separators is given in [5]: If both G and G^2 are chordal then there is a common perfect elimination ordering of these graphs (see also [3]). The existence of a common

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perfect elimination ordering of all chordal powers of an arbitrary given graph was proved in [2]. Such a common ordering can be computed in time $O(|V||E|)$ using a generalized version of maximum cardinality search which simultaneously uses chordality of these powers.

It is well-known that lexicographic breadth-first-search (LexBFS, [12]) and maximum cardinality search (MCS, [14]) give a perfect elimination ordering of a chordal graph in linear time.

In this paper we consider the question whether these algorithms working only on an initial chordal graph G produce a common perfect elimination ordering of chordal powers of G. We prove that every LexBFS-ordering of a chordal graph G gives a common perfect elimination ordering of all odd powers of G and characterize those chordal graphs by forbidden isometric subgraphs for which every LexBFS-ordering of the graph is a common perfect elimination ordering of all powers. The same questions we consider for MCS-orderings on chordal graphs.

2. Preliminaries

Throughout this paper all graphs $G = (V,E)$ are finite, undirected, simple (i.e. loop-free and without multiple edges) and connected.

A *path* is a sequence of vertices $v_0,\ldots,v_k$ such that $v_iv_{i+1} \in E$ for $i = 0,\ldots,k-1$; its *length* is k. A graph G is *connected* iff for every pair of vertices of G there is a path in G joining both vertices.

The *distance* $d_G(u,v)$ of vertices u,v is the minimal length of every path connecting these vertices. Obviously, d_G is a metric on G. If no confusion can arise we will omit the index G.

The *kth neighbourhood* $N^k(v)$ of a vertex v of G is the set of all vertices of distance k to v, i.e.

$$N^k(v) := \{u \in V\colon\ d_G(u,v) = k\},$$

whereas the *disk* of radius k centered at v is the set of all vertices of distance at most k to v:

$$D_G(v,k) := \{u \in V\colon\ d_G(u,v) \leqslant k\} = \bigcup_{i=0}^{k} N^i(v).$$

For convenience we will write $N(v)$ instead of $N^1(v)$. Again, if no confusion can arise we will omit the index G. The *kth power* G^k of G is the graph with the same vertex set V where two vertices are adjacent iff their distance is at most k.

Next we recall the definition and some characterizations of chordal graphs. An *induced cycle* is a sequence of vertices $v_0,\ldots,v_k$ such that $v_0 = v_k$ and $v_iv_j \in E$ iff $|i-j| = 1$ (modulo k). The *length* $|C|$ of a cycle C is its number of vertices. A graph G is *chordal* iff every induced cycle of G is of length at most three. One of the first results on chordal graphs is the characterization via dismantling schemes. A vertex v

of G is called *simplicial* iff $D(v,1)$ induces a complete subgraph of G. A *perfect elimination ordering* is an ordering of G such that v_i is simplicial in $G_i := G(\{v_i,\dots,v_n\})$ for each $i = 1,\dots,n$. It is well-known that a graph is chordal if and only if it has a perfect elimination ordering (cf. [8]). Moreover, there are two linear time algorithms for computing perfect elimination orderings of chordal graphs: lexicographic breadth-first-search (LexBFS, [8]) and maximum cardinality search (MCS, [14]). To make the paper self-contained we present the rules of these algorithms.

Let $s_1 = (a_1,\dots,a_k)$ and $s_2 = (b_1,\dots,b_l)$ be vectors of positive integers. Then s_1 is *lexicographically smaller* than $s_2 (s_1 < s_2)$ iff

1. there is an index $i \leqslant \min\{k,l\}$ such that $a_i < b_i$ and $a_j = b_j$ for all $j = 1,\dots,i-1$, or
2. $k < l$ and $a_i = b_i$ for all $i = 1,\dots,k$.

If $s = (a_1,\dots,a_k)$ is a vector and a is some positive integer then $s + a$ denotes the vector $(a_1,\dots,a_k,a)$.

procedure LexBFS
    Input: A graph $G = (V,E)$.
    Output: A LexBFS-ordering $\sigma = (v_1,\dots,v_n)$ of V.
    begin forall $v \in V$ **do** $l(v) := ()$;
      for $n := |V|$ **downto 1 do**
        choose a vertex $v \in V$ with lexicographically maximal label $l(v)$;
        define $\sigma(n) := v$;
        for all $u \in V \cap N(v)$ **do** $l(u) := l(u) + n$;
        $V := V \setminus \{v\}$;
      endfor;
  end.

procedure MCS
    Input: A graph $G = (V,E)$.
    Output: A MCS-ordering $\sigma = (v_1,\dots,v_n)$ of V.
      begin for $n := |V|$ **downto 1 do**
        choose a vertex $v \in V$ with a maximal number of numbered neighbours;
        number v by n;
        $\sigma(n) := v$;
        $V := V \setminus \{v\}$;
      endfor;
  end.

In the sequel we will write $x < y$ whenever in a given ordering of the vertex set of a graph G vertex x has a smaller number than vertex y. Moreover, $x < \{y_1,\dots,y_k\}$ is an abbreviation for $x < y_i$, $i = 1,\dots,k$.

In what follows, we will often use the following properties (cf. [9] for the first two):

(P_1) If $a < b < c$ and $ac \in E$ and $bc \notin E$ then there exists a vertex d such that $c < d$, $db \in E$ and $da \notin E$.

(P_2) If $a < b < c$ and $ac \in E$ and $bc \notin E$ then there exists a vertex d such that $b < d$, $db \in E$ and $da \notin E$.

(P_3) If $a < b < \{c_1, \dots, c_k\}$, $c_1, \dots, c_k$ pairwise distinct vertices, and $ac_i \in E$ and $bc_i \notin E$, $i = 1, \dots, k$, then there are pairwise distinct vertices, $d_1, \dots, d_k$ such that $b < d_i$, $d_i b \in E$ and $d_i a \notin E$, $i = 1, \dots, k$.

Evidently, (P_2) is a relaxation of both (P_1) and (P_3).

Lemma 2.1. (1) *Every LexBFS-ordering has property* (P_1).

(2) *Every ordering fulfilling* (P_1) *can be generated by LexBFS.*

(3) *Every MCS-ordering has property* (P_3).

(4) *Every ordering fulfilling* (P_3) *can be generated by MCS.*

Proof. (1) We refer to the well-known proof in [8].

(2) Let $\sigma = (v_1, \dots, v_n)$ be an ordering fulfilling (P_1) and suppose that $(v_{i+1}, \dots, v_n)$, $i \leqslant n-1$, can be produced by LexBFS but not $(v_i, \dots, v_n)$, i.e. v_i cannot be chosen via LexBFS. Let u be the vertex chosen next by LexBFS. Then there must be a vertex $w > v_i$ adjacent to u but not to v_i. We can choose w rightmost in σ. Thus in σ we have $u < v_i < w$, $uw \in E$ and $wv_i \notin E$. Now (P_1) implies the existence of a vertex $z > w$ adjacent to v_i but not to u. Since w is chosen rightmost all vertices with a greater number than w which are adjacent to u are adjacent to v_i too. Hence the LexBFS-label of v_i is greater than that of u, a contradiction.

(3) This follows directly from the rules of MCS.

(4) Let $\sigma = (v_1, \dots, v_n)$ be an ordering fulfilling (P_3) and suppose that $(v_{i+1}, \dots, v_n)$, $i \leqslant n-1$, can be produced by MCS but not $(v_i, \dots, v_n)$, i.e. v_i cannot be chosen via MCS. Let u be the vertex chosen next by MCS. By the rules of MCS we conclude $|N_{G_{i+1}}(u)| > |N_{G_{i+1}}(v_i)|$. In particular, $|P(u)| > |P(v_i)|$ where $P(u) := N_{G_{i+1}}(u) \setminus N_{G_{i+1}}(v_i)$ and $P(v_i) := N_{G_{i+1}}(v_i) \setminus N_{G_{i+1}}(u)$. Since $u < v_i$ in σ applying (P_3) to $u < v_i < P(u)$ yields $|P(v_i)| \geqslant |P(u)|$, a contradiction. □

An induced subgraph H of G is an *isometric* subgraph of G iff the distances within H are the same as in G, i.e.

$$\forall x, y \in V(H)\colon\ d_H(x, y) = d_G(x, y).$$

A set $S \subseteq V$ is *m-convex* (monophonically convex) iff for all pairs of vertices x, y of S each vertex of every induced path connecting x and y is contained in S too.

Lemma 2.2 (Farber and Jamison [7]). *If G is a chordal graph and $(v_1, \dots, v_n)$ is a perfect elimination ordering of G then G_i is m-convex and in particular an isometric subgraph of G for every $i = 1, \dots, n$.*

Thus, we conclude that $G^k(\{v_i, \dots, v_n\}) = G(\{v_i, \dots, v_n\})^k$ for every $i = 1, \dots, n$ and $k \in \mathbb{N}$. In the sequel we will often use m-convexity and isometricity of G_i in G.

Let $r : V \to \mathbb{N}$ be some vertex function defined on G. Then a set $D \subseteq V$ *r-dominates* G iff for all vertices x in $V \backslash D$ there is a vertex $y \in D$ such that $d(x,y) \leqslant r(x)$. D is a *r-dominating clique* iff D is complete and r-dominates G. Note that there are graphs and vertex functions r such that G has no r-dominating clique. For some graph classes there is an existence criterion for r-dominating cliques. Here we present it for chordal graphs.

Theorem 2.3 (Dragan and Brandstädt [4]). *Let G be a chordal graph and $r : V \to \mathbb{N}$. Then G has a r-dominating clique if and only if*

$$\forall u, v \in V\colon\ d(u,v) \leqslant r(u) + r(v) + 1.$$

Lemma 2.4. *Let G be a chordal graph and v, w, z be vertices of G pairwise at distance $l \geqslant 3$. Then there is a neighbour u of v of distance $l-1$ to both w and z.*

Proof. For proving the assertion we use the above existence theorem for r-dominating cliques in chordal graphs.

First consider the case $l = 2k-1$, $k \geqslant 2$. Define $r(v) = r(w) = r(z) = k-1$ and $r(x) = |V|$ for all remaining vertices. Then G has a r-dominating clique $\{c_1, c_2, c_3\}$ such that $d(v,c_1) = d(w,c_2) = d(z,c_3) = k-1$. By choosing shortest paths between v and c_1, w and c_2, and z and c_3, respectively, we obtain an isometric subgraph of G. Obviously, the neighbour u of v on a shortest path to c_1 fulfills $d(u,w) = d(u,z) = 2k-2$.

Now let $l = 2k$, $k \geqslant 2$. We define $r(v) = k-1$, $r(w) = r(z) = k$ and $r(x) = |V|$ for all remaining vertices, and obtain a minimum r-dominating clique C of size two or three. Moreover, there is exactly one vertex c in C at distance $k-1$ to v. Note that $d(c,w) = d(c,z) = k+1$. Again, the neighbour u of a shortest path between v and c fulfills the assertion. □

Lemma 2.5. *Let G be a chordal graph and v, w, z be vertices of G such that $d(w,z) = 2k+1$ and $d(v,w) = d(v,z) = 2k$, $k \geqslant 2$. Then there is a neighbour u of v of distance $2k-1$ to both w and z.*

Proof. We define $r(v) = k-1$, $r(w) = r(z) = k$ and $r(x) = |V|$ for all remaining vertices, and obtain a r-dominating clique C of size three. The neighbour u of a shortest path between v and vertex $c \in C$ r-dominating v fulfills the assertion. □

3. LexBFS-orderings

3.1. Odd powers of chordal graphs

At first we consider odd powers of chordal graphs. For technical reasons we handle the cube separately.

Lemma 3.1. *Every LexBFS-ordering of a chordal graph G is a perfect elimination ordering of G^3.*

Proof. Let $G = (V, E)$ be a chordal graph and $v \in V$ be the first vertex of an arbitrary LexBFS-ordering of G. Assume v is not simplicial in G^3. Then there must be vertices x, y in $D(v, 3)$ such that $d(x, y) \geqslant 4$. Since v is simplicial in G all vertices of $N^2(v)$ are pairwise of distance at most 3. Thus either x and y are both in $N^3(v)$ or, say, $x \in N^3(v)$ and $y \in N^2(v)$.

Case 1: $x \in N^3(v)$ and $y \in N^2(v)$. Choose vertices a, b, z as shown in Fig. 1. By distance requirements the subgraph induced by $\{v, a, b, z, x, y\}$ is isometric in G.

First assume $a < b$. From the m-convexity we immediately conclude $x < z < a < b$. Now we can apply (P_1) to the triple $v < x < a$ obtaining vertex $t > a$ which is adjacent to x but not to v. Since x is the smallest vertex in the path $t - x - z$ vertex t must be adjacent to z by m-convexity. The same argument can be applied now to the path $t - z - a$ implying $ta \in E$. Since $d(x, y) = 4$ and $tx \in E$ we have $tb \notin E$. From $a < b$ and $a < t$ the induced path $t - a - b$ yields a contradiction to the m-convexity.

Now let $b < a$. By the same arguments as above we obtain $y < b < a$ and the existence of a vertex w such that $w > a$, $wv \notin E$ and $\{y, b, a\} \subseteq N(w)$. From $d(x, y) = 4$ we conclude $wz \notin E$. Thus, the m-convexity with respect to the induced path $w - a - z$ implies $x < z < a$. As above we obtain a vertex $t > a$ adjacent to x, z, a but not to v, b, y, w. But now both endpoints of the induced path $t - a - w$ are greater than the mid-point, a contradiction to the m-convexity. This settles Case 1.

Case 2: $\{x, y\} \subseteq N^3(v)$. Choose vertices a, b, z, u as shown in Fig. 2. Note that $d(x, u) = d(z, y) = 3$ (and hence $d(x, y) = 4$) for otherwise we may apply Case 1. The graph induced by $\{v, a, b, z, u, x, y\}$ may contain the edges zb or au.

Case 2.1: $zb \in E$ or $au \in E$. We may assume $a = b$. W.l.o.g. let $z < u$. We conclude $x < z < a$ and apply (P_1) to the triple $v < x < a$ yielding a vertex $t > a$ which is adjacent to x but not to v. The m-convexity implies $tz \in E$ and $ta \in E$. By distance requirements we have $tu \notin E$. Thus m-convexity and the induced path $t - a - u$ imply

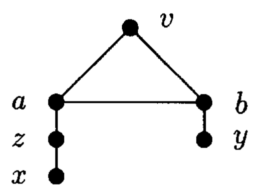

Fig. 1.

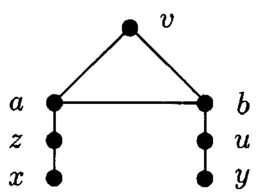

Fig. 2.

$y < u < a < t$. By similar arguments there is a vertex $s > a$ adjacent to y, u, a but not to v, z, x, t. The induced path $s - a - t$ contradicts the m-convexity.

Case 2.2: $zb \notin E$ and $au \notin E$. By symmetry we may assume $a < b$ implying $x < z < a < b$. (P_1) applied to $v < x < b$ gives a vertex $t > b$ adjacent to x but not to v. Now m-convexity implies (in this order) $tz, ta, tb \in E$. By distance requirements t cannot be adjacent to u. Thus $u < b$ and hence $y < u < b$. We can apply (P_1) to $v < y < b$ and obtain a vertex $s > b$ adjacent exactly to y, u, b. But now the induced path $t - b - s$ contradicts to the m-convexity. This settles the proof.

Now we can proceed by induction using isometricity of G_i in G by Lemma 2.2. □

Theorem 3.2. *For a chordal graph G every LexBFS-ordering of G is a perfect elimination ordering of each odd power G^{2k+1} of G.*

Proof. We prove the assertion by induction on k. For $k = 0$ the result is well-known, for $k = 1$ we are done by Lemma 3.1. So let $k \geqslant 2$, v be the first vertex of a LexBFS-ordering σ of G and assume that v is not simplicial in G^{2k+1}. Thus there must be vertices x, y in $D(v, 2k+1)$ such that $d(x, y) \geqslant 2k+2$. By the induction hypothesis v is simplicial in G^{2k-1}. Thus, every pair of vertices within $D(v, 2k-1)$ is at distance at most $2k-1$. Therefore neither x nor y are within the disk $D(v, 2k-1)$. Moreover, not both vertices x, y are in $N^{2k}(v)$. So we distinguish between two cases:

Case 1: $x \in N^{2k}(v)$ and $y \in N^{2k+1}(v)$. Choose arbitrary vertices $a \in N(x) \cap N^{2k-1}(v), b \in N^2(y) \cap N^{2k-1}(v)$ and a vertex $c \in N(y) \cap N^{2k}(v)$ which is rightmost in σ. The following distance equalities are easy to verify:

$$d(a,b) = 2k-1, \quad d(a,c) = d(b,x) = 2k,$$

$$d(a,y) = d(c,x) = 2k+1 \quad \text{and} \quad d(x,y) = 2k+2.$$

By applying Lemma 2.5 to the vertices v, x, c we obtain a neighbour u of v at distance $2k-1$ to both x and c. Let j denote the position of u in σ. By the induction hypothesis u is simplicial in $(G_j)^{2k-1}$, i.e. every pair of vertices within $D_{G_j}(u, 2k-1)$ is at distance at most $2k-1$. Since $d(c,x) = 2k+1$ not both vertices can be contained in G_j, i.e. $x < u$ or $c < u$.

First assume $x < u < c$ and consider a shortest path $u - w_1 - \cdots - w_{2k-2} - x$. Since, the path is chordless we conclude $x < w_{2k-2}$ for otherwise m-convexity implies that $u < w_1 < \cdots < w_{2k-2} < x < u$, a contradiction. Now applying (P_1) to $v < x < u$ yields a vertex $t > u$ adjacent to x but not to v. From $t > x$ and $w_{2k-2} > x$ we infer $tw_{2k-2} \in E$. Thus, $d(u,t) \leqslant 2k-1$. But now both t and c are in $D_{G_j}(u, 2k-1)$ implying $d(t,c) \leqslant 2k-1$. So we obtain $d(x,y) \leqslant d(x,t) + d(t,c) + d(c,y) \leqslant 2k+1$, a contradiction.

Now let $c < u$. Consider a shortest path $u - z_1 - \cdots - z_{2k-2} - c - y$. By the same argument as above $c < u$ implies $y < c < z_{2k-2}$. Now we apply (P_1) to $v < y < u$ obtaining a vertex $t > u$ adjacent to y. Note that $t \neq c$ since $c < u$. Then from

m-convexity we conclude $tc \in E$ and $tz_{2k-2} \in E$. Thus replacing c by $t > c$ is a contradiction to the choice of c.

Case 2: All vertices $x, y \in D(v, 2k+1)$ fulfilling $d(x,y) \geqslant 2k+2$ are contained in $N^{2k+1}(v)$. Choose arbitrary vertices $a_1 \in N^2(x) \cap N^{2k-1}(v)$, $a_2 \in N^2(y) \cap N^{2k-1}(v)$ and vertices $b_1 \in N(x) \cap N^{2k}(v)$, $b_2 \in N(y) \cap N^{2k}(v)$ which are rightmost in σ. Note $d(b_1, y), d(b_2, x) \neq 2k+2$. Since v is simplicial in G^{2k-1} we have $d(a_1, a_2) \leqslant 2k-1$. From $2k+2 \leqslant d(x,y) \leqslant 4 + d(a_1, a_2)$ we conclude $d(a_1, a_2) \geqslant 2k-2$. Moreover, $2k+2 \leqslant d(x,y) \leqslant 1 + d(x, b_2)$ implies $d(x, b_2) = d(y, b_1) = 2k+1$. Thus $d(x,y) = 2k+2$. Finally $d(b_1, b_2) \leqslant 2 + d(a_1, a_2)$ and $2k+2 = d(x,y) \leqslant 2 + d(b_1, b_2)$ gives $2k \leqslant d(b_1, b_2) \leqslant 2k+1$.

Case 2.1: $d(a_1, a_2) = 2k-1$. We apply Lemma 2.4 to the vertices v, a_1, a_2 and obtain a neighbour u of v at distance $2k-2$ to both a_1 and a_2. Thus $d(u, b_1) = d(u, b_2) = 2k-1$. Let j denote the position of u in σ. By induction hypothesis u is simplicial in G_j^{2k-1}. Since $d(b_1, b_2) \geqslant 2k$ not both vertices can be contained in $D_{G_j}(u, 2k-1)$. W.l.o.g. let $b_1 < u$. Consider a shortest path $u - w_1 - \cdots - w_{2k-2} - b_1 - x$. From $b_1 < u$ we infer $b_1 < w_{2k-2}$ implying $x < b_1 < w_{2k-2}$. Now we apply (P_1) to $v < x < u$ obtaining a vertex $t > u$ adjacent to x. Note that $t \neq b_1$ since $b_1 < u$. Then from m-convexity we conclude $tb_1 \in E$ and $tw_{2k-2} \in E$. Thus, replacing b_1 by $t > b_1$ is a contradiction to the choice of b_1.

Case 2.2: $d(a_1, a_2) = 2k-2$. We immediately conclude $d(b_1, b_2) = 2k$. Hence applying Lemma 2.4 to v, b_1, b_2 yields a neighbour u of v at distance $2k-1$ to both b_1 and b_2. Now proceed as in Case 2.1. □

Corollary 3.3. *A graph G is chordal if and only if every LexBFS-ordering of G is a common perfect elimination ordering of all odd powers of G.*

Note that we do not use chordality of odd powers in the above proofs. Thus, we can conclude:

Corollary 3.4. *Odd powers of chordal graphs are chordal.*

3.2. *Even powers of chordal graphs*

Now we consider even powers of chordal graphs which are in general not chordal.

Lemma 3.5. *If the first vertex v of a LexBFS-ordering of a chordal graph G is not simplicial in G^2 then G contains an isometric subgraph isomorphic to one of the graphs of Fig.* 3.

Proof. Let $G = (V, E)$ be a chordal graph and $v \in V$ be the first vertex of an arbitrary LexBFS-ordering of G. Assume v is not simplicial in G^2. Then there must be vertices x, y in $D(v, 2)$ such that $d(x,y) \geqslant 3$. We may choose x, y rightmost in the LexBFS-ordering. Since v is simplicial in G we immediately have $d(x,y) = 3$. Choose vertices

F_1 F_2

Fig. 3.

$a < b$ in $N(v)$ such that $ax, by \in E$, $ay, bx \notin E$ and both are rightmost in the LexBFS-ordering. The m-convexity then implies $x < a < b$. Thus, we can apply (P_1) to $v < x < b$ obtaining a vertex $u > b$ adjacent to x but not to v. Again m-convexity gives the edges ua and ub. From $d(x, y) = 3$ we infer $uy \notin E$ implying $y < b$.

Case 1: $x < y < b$. Then (P_1) applied to $x < y < u$ yields a vertex $w > u$ adjacent to y but not to x. By m-convexity the path $b - y - w$ cannot be induced, hence $bw \in E$. The same argument applied to $u - b - w$ gives $uw \in E$. Suppose $vw \in E$. Then the simplicity of v implies $wa \in E$ and we can replace b by $w > b$ contradicting the maximality of b. Thus $vw \notin E$ and we obtain either F_1 or F_2.

Case 2: $y < x < a < b$. By applying (P_1) to $v < y < a$ we obtain a vertex $w > a$ adjacent to y but not to v. Note that we can choose w to be rightmost in the LexBFS-ordering. Since the path $b - y - w$ cannot be induced by m-convexity we have $bw \in E$. If $uw \in E$ or $b < w$ (which implies $uw \in E$) then we have either graph F_1 or F_2. So let $w < b$ and $uw \notin E$. Hence, $aw \notin E$, for otherwise $a < u$ and $a < w$ imply $uw \in E$. Since $d(x, y) = 3$, $w > y$ and by the rightmost choice of x, y we conclude $d(x, w) = 2$. Let z be a vertex adjacent to both x and w. By chordality $za, zb \in E$. If $zv \notin E$ then the vertices $\{v, a, b, x, w, y, z\}$ induce an isometric subgraph isomorphic to F_1. Otherwise, from the choice of a we infer $z < a$. But now $aw \notin E$ implies $\{z, w\} > a > z$, a contradiction. □

Corollary 3.6. *Let G be a chordal graph. Then LexBFS produces for every induced subgraph H of G a perfect elimination ordering in H^2 if and only if G does not contain the graphs of Fig.* 3 *as induced subgraphs.*

Proof. In Fig. 3 valid LexBFS-orderings of the graphs are given which are not perfect elimination orderings in the square. □

Theorem 3.7. *If G is a chordal graph which does not contain the graphs of Fig.* 3 *as isometric subgraphs then every LexBFS-ordering of G is a perfect elimination ordering of each even power G^{2k}, $k \geqslant 1$, of G.*

Proof. We prove the assertion by induction on k. For $k = 1$ we are done by Lemma 2.2 and by Lemma 3.5. So let $k \geqslant 2$ and assume that the first vertex v of a LexBFS-ordering σ of G is not simplicial in G^{2k} but in $G, \dots, G^{2k-1}$. Then there must be vertices $x, y \in D(v, 2k)$ such that $d(x, y) \geqslant 2k + 1$. Since v is simplicial in G^{2k-1} we immediately conclude that

$$x, y \in N^{2k}(v), \quad d(x,y) = 2k+1 \quad \text{and} \quad d(a,b) = 2k-1$$

where $a \in N(x) \cap N^{2k-1}(v)$ and $b \in N(y) \cap N^{2k-1}(v)$ are rightmost in σ. Thus, we can apply Lemma 2.4 to v, a, b obtaining a neighbour u of v at distance $2k-2$ to both a and b. Let j denote the position of u in σ. By induction hypothesis u is simplicial in G_j^{2k-2}. Thus, $d(a,b) = 2k-1$ implies $a < u$ or $b < u$. W.l.o.g. let $a < u$. Consider a shortest path $u - w_1 - \cdots - w_{2k-3} - a - x$. From $a < u$ we infer $a < w_{2k-3}$ implying $x < a < w_{2k-3}$. Now we apply (P_1) to $v < x < u$ obtaining a vertex $t > u$ adjacent to x. Note that $t \neq a$ since $a < u$. Then from m-convexity we conclude $ta \in E$ and $tw_{2k-3} \in E$. Thus replacing a by $t > a$ is a contradiction to the choice of a. □

Corollary 3.8. *A graph G is chordal and does not contain the graphs of Fig.* 3 *as isometric subgraphs if and only if every LexBFS-ordering of G is a perfect elimination ordering of G and of each even power G^{2k}, $k \geqslant 1$, of G.*

Proof. If G is chordal and does not contain the graphs of Fig. 3 as isometric subgraphs then we are done by Theorem 3.7.

To prove the converse first note that G is chordal since G has a perfect elimination ordering. Assume that G contains one of the graphs of Fig. 3 as an isometric subgraph, say F_1. We start LexBFS with the vertex labeled by 7 in Fig. 3 yielding label n. Now we may label vertex 6 by $n-1$ and vertex 5 by $n-2$. Let k, l, s, t be the labels of vertices 4, 3, 2, 1, respectively. By the rules of LexBFS t must be the smallest label among k, l, s, t. Thus vertex t is not simplicial in G_t^2 since $d(l,s) = 3$ in the isometric subgraph G_t of G, a contradiction. For F_2 we can proceed in a similar way. □

Corollary 3.9. *A graph G is chordal and does not contain the graphs of Fig.* 3 *as isometric subgraphs if and only if every LexBFS-ordering of G is a perfect elimination ordering of each power G^k, $k \geqslant 1$, of G.*

Corollary 3.10. *If G is chordal and does not contain the graphs of Fig.* 3 *as isometric subgraphs then all powers of G are chordal.*

Corollary 3.11. *If T is a tree then every LexBFS-ordering of T is a common perfect elimination ordering of all powers of T.*

4. MCS-orderings

In this section we characterize those chordal graphs G for which every MCS-ordering of all induced subgraphs H of G is a common perfect elimination ordering of all powers

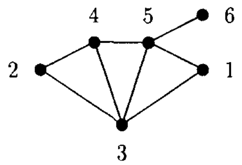

Fig. 4.

of H. As we will show even for trees MCS does not give a common perfect elimination ordering of all powers.

Lemma 4.1. *For every induced subgraph H of a chordal graph G MCS produces a perfect elimination ordering of H^2 if and only if G does not contain the graph of Fig. 4 as induced subgraph.*

Proof. In Fig. 4 a MCS-ordering is given which is not a perfect elimination ordering of the square.

The converse we prove by assuming the contrary. Let v be the first vertex of a MCS-ordering of a chordal graph G and suppose v is not simplicial in G^2. Then there must be two vertices x, y in $D(v,2)$ of distance at least 3. Since v is simplicial in G there must be (adjacent) vertices a, b in $N(v)$ such that $ax, by \in E$ but $ay, bx \notin E$. W.l.o.g. we may assume $a < b$. Moreover, we may choose x, y such that the sum of their numbers in the MCS-ordering is as large as possible.

From m-convexity we immediately obtain $x < a < b$ by considering the induced path $x - a - b$. Thus, we can apply (P_2) to $v < x < b$ obtaining a vertex $u > x$ adjacent to x but not to v. Note that by distance requirements u is not adjacent to y. Since both endpoints of the path $u - x - a$ are greater than the mid-point we have $ua \in E$ by m-convexity. If $ub \in E$ then we are done. So let $ub \notin E$. By the choice of x, y we have $d(u, y) = 2$ for otherwise we can replace x by $u > x$. Let w be a vertex adjacent to u and y. By considering the 5-cycle $w - u - a - b - y - w$ the chordality of G implies that $wa, wb \in E$. Denote $F := G(\{v, a, b, x, y, u, w\})$. If $wv \in E$ then $F\backslash\{b\}$ is isomorphic to the graph of Fig. 4. Otherwise $F\backslash\{u\}$ gives the desired graph. $\square$

Note that (P_2) is not sufficient to obtain the results of the next lemma: In Fig. 5 we present a chordal graph with an ordering satisfying (P_2) which cannot be produced by MCS. Observe that the vertex numbered by 1 is not simplicial in the cube.

Lemma 4.2. *For every induced subgraph H of a chordal graph G MCS produces a perfect elimination ordering of H^3 if and only if G does not contain the graphs of Fig. 6 as induced subgraphs.*

Proof. It is easy to verify that the MCS-orderings of the graphs given in Fig. 6 are not perfect elimination orderings of the cubes.

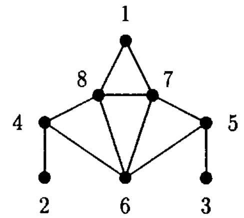

Fig. 5. An ordering satisfying (P_2) which cannot be produced by MCS.

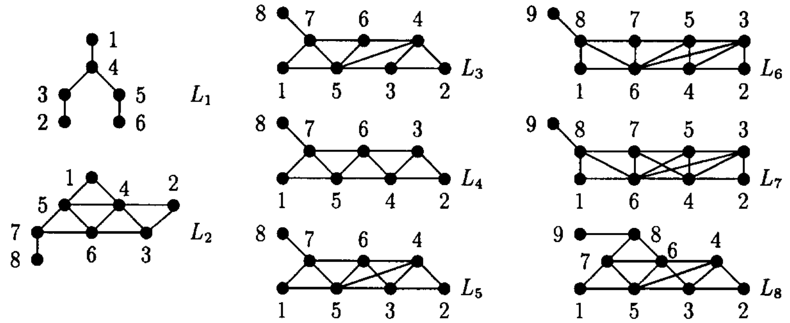

Fig. 6.

Let v be the first vertex of a MCS-ordering σ and assume that v is not simplicial in G^3. Note that v is simplicial in G. Thus there are vertices $x, y \in D(v,3)$ such that $d(x,y) \geqslant 4$. Analogously to the proofs for LexBFS-orderings $d(x,y) = 4$ and either $x, y \in N^3(v)$ or $x \in N^2(v)$ and $y \in N^3(v)$.

Case 1: $x \in N^2(v)$ and $y \in N^3(v)$. Choose vertices $a, b \in N(v)$, $a \neq b$, and $c \in N^2(v)$ such that $ax \in E$, $bc \in E$, $cy \in E$ and such that the sum τ of the positions of the vertices $\{x, y, a, b, c\}$ in σ is maximal.

Case 1.1: $b < a$. By m-convexity we must have $y < c < b < a$. Applying (P_3) to $v < y < \{a, b\}$ yields vertices w_1, w_2 such that $w_i > y, w_i y \in E$ and $w_i v \notin E$, $i = 1,2$. By m-convexity $d(w_i, c) \leqslant 1$, $i = 1,2$. We may choose w_1, w_2 rightmost in σ. If for one vertex $w_i, i = 1,2, c < w_i$ holds then by m-convexity $w_i b \in E$. Thus we can replace c by w_i which is a contradiction to the maximality of τ. Hence, $w_i \leqslant c$ for both $i = 1,2$. Since c is a feasible choice we conclude $\{w_1, w_2\} = \{w, c\}$, $w < c$ and $wc \in E$. Now applying (P_3) to $v < c < a$ gives a vertex $t > c$ adjacent to c but not to v. We choose t rightmost in σ. Since, $tv \notin E$ we have $t \neq b$, and by m-convexity t is adjacent to b. From $tc \in E$ we conclude $tx \notin E$. If $ty \in E$ then replacing c by $t > c$ increases τ, a contradiction. Thus $ty \notin E$.

Case 1.1.1: $wb \in E$. If $ta \in E$ then we obtain either L_3 or L_5 depending on whether $tw \in E$ or not. So let $ta \notin E$. Hence, $t < b$ for otherwise $ta \in E$ by m-convexity. Now $v < y < w < c < t < b < a$. If $tw \notin E$ then $\{t, b, w, y, a, x\}$ induces a graph isomorphic to L_1. Otherwise applying (P_3) to $v < t < a$ gives a vertex $s > t$ adjacent to t but not to v. By m-convexity and $t < b$ we have $sb \in E$. Assume $sc \in E$. Then we can replace t by $s > t$, a contradiction to the choice of t. Thus $sc \notin E$. Supposing $sy \in E$ we can replace c by $s > c$ increasing τ, again a contradiction. Therefore, $sy \notin E$. If $sx \in E$ then $sw \notin E$ and so $\{v,b,s,x,w,y\}$ induces a graph isomorphic to L_1. Now let $sx \notin E$. If $sa \notin E$ then $\{s,b,c,y,a,x\}$ induces a graph isomorphic to L_1. Otherwise we obtain L_6 or L_7 depending on the adjacency of s and w.

Case 1.1.2: $wb \notin E$. Since τ is maximal we may not replace y by w. Hence either $d(w,v) = 2$ or $d(w,x) = 3$. In the former case there must be a common neighbour q of v and w different from a and b. By simplicity of v we have $qa \in E$ and thus $d(w,x) = 3$. Therefore, we have only to consider the second case.

First consider the case $d(w,a) = 2$ and let z be a common neighbour of w and a. Consider the cycle $a-b-c-w-z-a$. By chordality and $wa \notin E$ we immediately conclude $zb,zc \in E$. By distance requirements we have $zx,zy \notin E$. Thus, without the vertex t, we get L_4 provided $zv \notin E$. Now let $zv \in E$. Then the maximality of τ implies $z < b$. Assume $ta \notin E$. If also $tw \notin E$ then for $tz \in E$ the set $\{t,z,a,x,w,y\}$ induces L_1, and for $tz \notin E$ the set $\{t,b,c,a,z,w,y,x\}$ induces L_2.

If $tw \in E$ then $tz \in E$ and we can proceed as in Case 1.1.1. by replacing b by z. Note that $t < z$ for otherwise $ta \in E$ by m-convexity, a contradiction.

Now assume $ta \in E$. Hence $zt \in E$ and $\{v,a,z,x,t,c,w,y\}$ induces a graph isomorphic to L_3 or L_5 depending on whether $tw \in E$.

So let $d(w,a) = 3$ and let $w-z_1-z_2-x$ be a shortest path between w and x. Note $az_1 \notin E$ and $cz_2 \notin E$. By chordality the cycle $a-b-c-w-z_1-z_2-x-a$ must contain chords. Obviously every such chord must be incident with z_1 or z_2. Thus, we conclude $az_2,bz_2,bz_1,cz_1 \in E$. If $vz_2 \notin E$ then $\{v,b,c,y,z_2,x\}$ induces a graph isomorphic to L_1, otherwise $\{v,z_2,b,x,c,w,z_1,y\}$ induces a graph isomorphic to L_4. Note that the simplicity of v and $az_1 \notin E$ imply $vz_1 \notin E$.

Case 1.2: $a < b$. By m-convexity we have $v < x < a < b$. Applying (P_3) to $v < x < b$ gives a (rightmost chosen) vertex $u > x$ adjacent to x but not to v. Thus $u \neq a$ and m-convexity gives $ua \in E$. By distance requirements we have $uc,uy \notin E$. If $ub \in E$ then $\{v,b,c,y,u,x\}$ induces a graph isomorphic to L_1. So let $ub \notin E$. We immediately conclude $u < a$ and $d(u,y) = 3$. First consider the case $d(u,c) = 2$ and let w be a common neighbour of u and c. By chordality we must have the chords wa and wb in the 5-cycle $a-b-c-w-u-a$. Moreover, $wx,wy \notin E$ by distance requirements. Thus, if $wv \notin E$ then we have a graph isomorphic to L_2, otherwise $\{v,w,c,y,u,x\}$ induces a graph isomorphic to L_1.

Now let $d(u,c) = 3$ and let $u-w_1-w_2-y$ be a shortest path between u and y. Note $w_1y,uw_2,aw_2,cw_1,vw_2 \notin E$. Again, chordality implies the chords aw_1,bw_1,cw_2,w_2b of the cycle $a-b-c-y-w_2-w_1-u-a$. If $vw_1 \notin E$ then $\{v,a,b,x,u,w_1,w_2,y\}$ induces a graph isomorphic to L_2, otherwise $\{v,w_1,w_2,y,u,x\}$ induces a graph isomorphic to L_1.

Case 2: All vertices $x,y \in D(v,3)$ fulfilling $d(x,y) \geqslant 4$ are contained in $N^3(v)$. Choose vertices $a \in N(v) \cap N^2(x)$, $b \in N(v) \cap N^2(y)$, $u \in N(a) \cap N(x)$ and $z \in N(b) \cap N(y)$ such that the sum τ of the positions of these vertices in σ is maximal. Since v is simplicial in G we have $ab \in E$. Note also that $d(x,z) = d(y,u) = 3$ and so $d(x,y) = 4$. If $ub \in E$ or $az \in E$ we immediately obtain the graph L_1 as induced subgraph. Thus $ub \notin E$ and $az \notin E$.

First we claim $d(u,z) = 2$. Assume $d(u,z) = 3$ and let $x - w_1 - w_2 - z$ be a shortest path between x and z. Hence, $w_1 \neq u$ and $w_1z, xw_2, uw_2 \notin E$. If $w_2 = b$ then we obtain an induced graph isomorphic to L_1. So $w_2 \neq b$. As before chordality implies the chords w_1u, w_1a, w_2a, w_2b of the cycle $a - b - z - w_2 - w_1 - x - u - a$. Note that $w_1b \notin E$ for otherwise $\{v, b, w_1, x, z, y\}$ induces a graph L_1. Now $\{v, w_2, w_1, x, z, y\}$ induces a graph L_1 if $vw_2 \in E$, and $\{v, a, b, u, z, w_1, w_2, y\}$ induces a graph L_2 if $vw_2 \notin E$.

So $d(u,z) = 2$, and let $w \in N(u) \cap N(z)$. From chordality of the graph we conclude $wa, wb \in E$. By distance requirements we have $wx, wy \notin E$. Since L_1 is forbidden $wv \notin E$.

Now w.l.o.g. we may assume $a < b$. Thus, m-convexity gives $v < x < u < a < b$. Applying (P_3) to $v < x < \{a, b\}$ gives a vertex t different from u such that $x < t$, $tx \in E$ and $tv \notin E$. Note that by distance requirements $tz, ty \notin E$, but by m-convexity $tu \in E$. Suppose $t > u$. Then $ta \notin E$ for otherwise replacing u by $t > u$ increases τ, a contradiction. Thus, $t - u - a$ is an induced path, but $u < \{t, a\}$ — a contradiction to m-convexity. Therefore, $t < u$. If $tb \in E$ then $\{v, b, z, y, t, x\}$ induces a graph isomorphic to L_1. If $tb, tw \notin E$ but $ta \in E$ then $\{v, a, w, z, t, x\}$ induces L_1. If $tb, ta \notin E$ but $tw \in E$ then $\{a, w, z, y, t, x\}$ induces the same graph L_1. When $tb \notin E$ but $ta, tw \in E$ we obtain L_8. So it remains to consider the case $ta, tb, tw \notin E$. Since, $t > x$ but τ is maximal we may not replace x by t. Thus, either $d(t,v) = 2$ or $d(t,y) = 3$. If $d(t,v) = 2$ then $d(t,y) \leqslant 3$ by the presumptions of this case. So we have $d(t,y) = 3$.

Case 2.1: $d(t,z) = 2$. Let s be a common neighbour of t and z. Then chordality implies the chords su and sw in the cycle $u - w - z - s - t - u$. The same argument applied to the cycle $a - b - z - s - u - a$ gives $sa, sb \in E$. Moreover, by distance requirements we have $sx, sy \notin E$. Thus $\{a, s, z, y, t, x\}$ induces a graph isomorphic to L_1.

Case 2.2: $d(t,z) = 3$. Consider a shortest path $t - s_1 - s_2 - y$. Since $us_2, zs_1 \notin E$ the chordality of the cycle $u - w - z - y - s_2 - s_1 - t - u$ implies the chords us_1, ws_1, ws_2, zs_2. Now we obtain the cycle $a - b - z - s_2 - s_1 - u - a$ implying the chords s_1a, s_2b and s_1b or s_2a. Note that $s_2v \notin E$ by $d(v,y) = 3$. If $s_2a \in E$ then $\{v, a, s_2, y, u, x\}$ induces a graph isomorphic to L_1. If $s_2a \notin E$ then $\{v, b, z, y, s_1, t\}$ induces L_1 for $s_1v \notin E$ or $\{v, s_1, s_2, y, t, x\}$ induces L_1 for $s_1v \in E$. □

Theorem 4.3. *For every induced subgraph H of a chordal graph G MCS produces a common perfect elimination ordering of each power H^k, $k \geqslant 1$, if and only if G does not contain the graphs of Fig. 7 as induced subgraphs.*

Proof. First observe that there are MCS-orderings of the graphs of Fig. 7 which are not perfect elimination orderings in the square for M_1, in the cube for L_1, L_8, and in the 4th power for M_2, respectively.

Now let G be a chordal graph which does not contain the graphs of Fig. 7 as induced subgraphs. Let σ be a MCS-ordering of G. We prove by induction on k that σ is a perfect elimination ordering in G^k. Since, the graphs $L_2, \dots, L_7$ of Fig. 6 contain M_1 we conclude from Lemmas 4.1 and 4.2 that σ is a perfect elimination ordering of G^2 and G^3. Suppose the first vertex v of σ is not simplicial in G^k, $k \geqslant 4$. Then

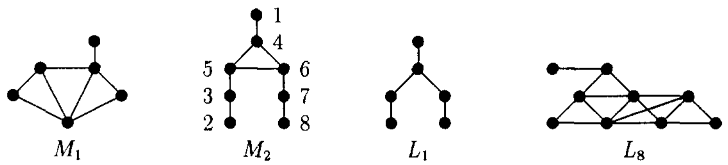

Fig. 7.

there must be vertices $x, y \in D(v,k)$ such that $d(x,y) \geqslant k+1$. Since, v is simplicial in G^{k-1} all vertices within $D(v,k-1)$ are at distance at most $k-1$. Thus, $x, y \in N^k(v)$, $d(x,y) = k+1$ and there are vertices $a,b \in N^{k-1}(v)$ such that $d(a,b) = k-1$, $ax \in E$ and $by \in E$.

If $k = 2l$ then define $r(v) = r(a) = r(b) = l-1$ and $r(u) = |V|$ for all remaining vertices of G. By Theorem 2.3 there is a r-dominating clique of G. Obviously, a minimum one has size three. Thus, we obtain graph M_2 as induced subgraph of G, a contradiction.

Now let $k = 2l+1$ and define $r(v) = l-1$, $r(a) = r(b) = l$ and $r(u) = |V|$ for all remaining vertices of G. By Theorem 2.3 there is a r-dominating clique of G. Either there is a vertex c at distance l to v, a, b — then we obtain L_1 as induced subgraph of G — or every minimum r-dominating clique is a triangle. Since, $d(x,y) = 2l$ and $l \geqslant 2$ now we obtain M_2 as induced subgraph of G. $\square$

5. Conclusions

We want to thank the anonymous referees of [2] for asking whether one can obtain a common perfect elimination ordering of chordal powers of a given graph by using the well-known linear time methods LexBFS or MCS. As this paper shows even for chordal powers of chordal graphs these algorithms do not give a common perfect elimination ordering. Moreover, note that the (nonlinear) method for producing lexical orderings [11] of graphs also does not give such an ordering: the labeling of the graph F_1 in Fig. 3 is a lexical ordering of this graph but not a perfect elimination ordering of its square [13].

On the other hand, as a consequence of the presented results, any LexBFS-ordering of a ptolemaic or interval graph (for definitions we refer to [1,8]) is a common perfect elimination ordering of all powers: all graphs of Fig. 3 contain induced subgraphs which are forbidden for these graphs.

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